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Real-time quantum field theory at finite temperature in an inhomogeneous media

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Abstract

The method of the real time perturbative calculations of nonequilibrium averages is generalised to the case of varying chemical potential. Calculations are performed in the frame of Zubarev's nonequilibrium density matrix approach. In this approach perturbations of temperature and other thermodynamical parameters are taken into account explicitly including nonlinear terms. It differs from the Schwinger-Keldysh approach through the choice of more general initial conditions for the density matrix.

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1 Introduction

Quantum field theory (QFT) at finite temperature in the equilibrium [1, 2, 3, 4, 5] and nonequilibrium [6, 7, 8, 9] cases has many applications in study of the early Universe [10, 11, 12] and heavy ion collisions [13]. Both kind of systems are intrinsically nonequilibrium in nature, and only some details of their behaviour may be considered in the frame of equilibrium description.

The purpose of this paper is to introduce a Green function generating functional for the nonequilibrium QFT at finite temperature and chemical potential in the case when these thermodynamical parameters depend on space and time coordinates.

The approach is based on Zubarev's nonequilibrium density matrix [14] (called by author nonequilibrium statistical operator (NSO) [15]). The NSO is a solution of the Liouville equation with initial conditions in the form of a local-equilibrium density matrix. This matrix does not describe irreversible processes such as thermal conductivity. It corresponds to the maximum entropy of a system for a given distribution of energy, charge density etc., but it is not a solution of the Liouville equation, that rules the system dynamics. This deficiency plays no role if the initial condition in the form of a local-equilibrium density matrix is fixed at the time $t \rightarrow -\infty$. Hydrodynamical equations which account for the dissipative terms are derived from the NSO explicitly (see [15]).

A situation with varying temperature is already analized in [16] and account of a chemical potential will be performed here in the same manner. However, there are some technical differences. It is shown in [16] that consideration of varying temperature requires the introduction of a more complicate time integration contour in the complex time integration plane in comparison with the equilibrium case [4]. It will be shown here that in the case of a uniform temperature distribution and varying chemical potential a complicated contour is not required. Thus, temperature plays a special role in the quantum field theoretical description.

There are other approaches to the study of QFT in inhomogeneous media. It was argued in [17], Zubarev's [14, 15] and McLeennan's [18, 19] approaches are the most useful, since they take into account thermal and charge inhomogeneities explicitly. These approaches have the same numerical results.

The most used method in QFT, that due to Schwinger-Keldysh [20] is also based on the Liouville equation but with equilibrium initial condition for the density matrix. Only those thermal perturbations that occur after strong external action on the initially equilibrium system are taken into account by this approach. It means that an inhomogeneous distribution of the initial values of thermal parameters is not taken into consideration. And in general, this approach does not determine an explicit expression for the temperature chemical potential and other thermodynamical parameters.

2 Density matrix

It is possible to consider nonequilibrium stationary systems in the frame of a generalised Gibbs method. The idea of local equilibrium is very useful for this aim. It is based on

the relaxation time property of nonequilibrium systems. With this property, the system rapidly achieves a local equilibrium state and then slowly relaxes to equilibrium [21]. Investigation of the second long stage is possible in terms of NSO. As it is shown in [15], the following local equilibrium density matrix corresponds to the maximum of the system entropy

$$\hat{\rho}_l(t) = \Xi_l^{-1} e^{-\int d^3\mathbf{x} \beta^\nu(\mathbf{x}, t) \hat{T}_{0\nu}(\mathbf{x}, t) + \int d^3\mathbf{x} \alpha(\mathbf{x}, t) \hat{J}_0(\mathbf{x}, t)}, \quad (2.1)$$

where

$$\beta^\nu(x) = \beta(x) u^\nu(x), \quad \alpha(x) = \beta(x) \mu(x), \quad (2.2)$$

$\beta(x) = 1/T(x)$ — inverse local temperature, $u^\nu(x)$ — hydrodynamical velocity, $\mu(x)$ — chemical potential corresponding to the charge density \hat{J}_0 (μ^k and J_0^k if there are some conserved charges in the system), $\hat{T}_{\mu\nu}$ is the energy momentum tensor and Ξ_l — the partition function that normalises (2.1). It is clear that in the limit of uniform values of temperature and chemical potential distribution we recover the well-known grand canonical Gibbs density matrix, that in the rest frame (where $u^\mu = (1, 0, 0, 0)$) is

$$\hat{\rho}_0 = \Xi_0^{-1} e^{-\beta \hat{H} + \alpha \hat{Q}_0}, \quad (2.3)$$

where \hat{H} is the Hamiltonian of the system and

$$\hat{Q}_0 = \int d^3\mathbf{x} \hat{J}_0(\mathbf{x}, t) \quad (2.4)$$

its charge.

The density matrix (2.1) gives the correct results for the energy and charge density. But it doesn't describe dissipative processes such as thermal conductivity, diffusion, viscosity etc.

A density matrix that completely describes the dynamics of the system including dissipative processes is obtained by Zubarev as a retarded solution of the Liouville equation [15]

$$\frac{\partial \hat{\rho}}{\partial t} - i[\hat{\rho}, \hat{H}] = 0 \quad (2.5)$$

with the initial condition in the form of a local-equilibrium density matrix (2.1). For this aim an external source that breaks time invariance of the equation is introduced in the right hand side of (2.5)

$$\frac{\partial \hat{\rho}_\varepsilon}{\partial t} - i[\hat{\rho}_\varepsilon, \hat{H}] = -\varepsilon(\hat{\rho}_\varepsilon - \hat{\rho}_l). \quad (2.6)$$

The limit $\varepsilon \rightarrow 0$ is to be taken after thermodynamical averaging. The same procedure of the retarded solution selection of the equation of motion with the free initial conditions is used in scattering theory [22]. The solution of (2.6) is [14, 15]

$$\hat{\rho}_\varepsilon(t) = \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} \hat{\rho}_l(t + t_1). \quad (2.7)$$

Thus, integration of (2.6) with the positive sign of ε gives the retarded solution of the Liouville equation (2.5). This solution corresponds to the entropy

$$S = -\text{Tr}\{\hat{\rho}_\varepsilon \ln \hat{\rho}_\varepsilon\} \quad (2.8)$$

increase. There are some other useful techniques to obtain the smooth density matrix describing entropy increase (see for example [15]). The exact solutions $\hat{\rho}_{exact}$ of (2.5) corresponds to the conserved entropy $S = -\text{Tr}\{\hat{\rho}_{exact} \ln \hat{\rho}_{exact}\}$. It may be averaged in quantum states with close energies or in time intervals to obtain coarse grained

$$\hat{\rho}_\Gamma(t) = \frac{1}{\Delta\Gamma} \text{Tr}_{\Delta\Gamma} \hat{\rho}_{exact}(t) \quad (2.9)$$

or time averaged

$$\hat{\rho}_\tau(t) = \frac{1}{T} \int_t^{t+\tau} dt' \hat{\rho}_{exact}(t') \quad (2.10)$$

density matrices. So, $S_\Gamma = -\text{Tr}\{\hat{\rho}_\Gamma \ln \hat{\rho}_\Gamma\}$ and $S_\tau = -\text{Tr}\{\hat{\rho}_\tau \ln \hat{\rho}_\tau\}$ increase in time.

In the general case, matrix (2.1) is expressed [23, 24]

$$\hat{\rho}_l = \Xi_l^{-1} e^{-\int d\sigma^\mu (\beta^\nu(\mathbf{x},t) \hat{T}_{\mu\nu}(\mathbf{x},t) - \alpha(\mathbf{x},t) \hat{J}_\mu(\mathbf{x},t))}, \quad (2.11)$$

where σ_μ defines some space-like hypersurface. Now it is possible to introduce transformed energy-momentum tensor depending on chemical potential.

$$\begin{aligned} \hat{\rho}_l &= \Xi_l^{-1} e^{-\int d\sigma^\mu (\beta^\nu(\mathbf{x},t) \hat{T}_{\mu\nu}(\mathbf{x},t) - \mu(\mathbf{x},t) \beta^\nu(\mathbf{x},t) u_\nu(\mathbf{x},t) \hat{J}_\mu(\mathbf{x},t))} = \\ &= \Xi_l^{-1} e^{-\int d\sigma^\mu \beta^\nu(\mathbf{x},t) \hat{T}'_{\mu\nu}(\mathbf{x},t)}, \end{aligned} \quad (2.12)$$

where

$$\hat{T}'_{\mu\nu}(x) = \hat{T}_{\mu\nu}(x) - \mu(x) u_\nu(x) \hat{J}_\mu(x), \quad (2.13)$$

and identification

$$\beta(x) \equiv \beta^\nu(x) u_\nu(x) \quad (2.14)$$

was used.

When the distance of the hydrodynamical parameter variations L is bigger than the correlation lengths

$$L \gg l_{corr} \quad (2.15)$$

the hypersurface σ^μ may be chosen in an arbitrary way [23] and it is possible to consider the hypersurface for which $dx^0 = 0$ ($d\sigma^\mu = (d^3\mathbf{x}, 0, 0, 0)$). In this case NSO may be written in the form (2.7)

$$\hat{\rho}_\varepsilon = \varepsilon \int_{-\infty}^0 dt_1 \Xi_l^{-1} e^{\int d^3\mathbf{x} \beta^\nu(\mathbf{x},t+t_1) \hat{T}'_{0\nu}(\mathbf{x},t+t_1)}, \quad (2.16)$$

with $\hat{T}'_{0\nu}$ defined by (2.13).

3 Thermal perturbations

Consider the inverse temperature distribution slightly differing from a certain constant value $\beta_0^\mu = \beta_0 u_0^\mu$ by a small parameter $\beta_1^\mu(x)$

$$\beta^\mu(x) = \beta_0^\mu + \beta_1^\mu(x). \quad (3.1)$$

Using this expression it is possible to apply our results [25, 16, 17] but with the transformed energy-momentum tensor (2.13) in the NSO (2.16). NSO may be written in the rest frame ($u_0^\mu = (1, 0, 0, 0)$). In this frame $\beta_0^\mu \hat{P}_\mu = \beta_0 \hat{H}$

$$\begin{aligned} \hat{\rho}_\varepsilon(t) = \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} \Xi_l^{-1} \exp & \left[-\beta_0 \hat{H}' - \right. \\ & \left. - \int d^3 \mathbf{x} \beta_1^\mu(\mathbf{x}, t + t_1) \hat{T}'_{0\mu}(\mathbf{x}, t + t_1) \right] \end{aligned} \quad (3.2)$$

where

$$\hat{H}' = \hat{H} - \int d^3 \mathbf{x} \mu(x) \hat{J}^0(x). \quad (3.3)$$

Now we make use of a well-known formula for the exponent of noncommuting operator sums [26] rewritten for our case in [25],

$$\exp \left[-\beta_0 \hat{H}' + \hat{B}(\tau) \right] = e^{-\beta_0 \hat{H}'} \mathbf{T}_C \left\{ \exp \left[i \int_C dx_0 \beta_0^{-1} \hat{B}(\tau + x_0) \right] \right\}, \quad (3.4)$$

where C is an arbitrary contour in the complex time-plane appropriate for the real time QFT [2, 4, 27, 28, 29] and \mathbf{T}_C — is an ordering by the time variable along C . With the help of (3.4), NSO may be presented as

$$\begin{aligned} \hat{\rho}_\varepsilon(t) = \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} \Xi_l^{-1} e^{-\beta_0 \hat{H}'} \times \\ \times \mathbf{T}_C \left\{ \exp \left[-i \int_C d^4 x \frac{\beta_1^\mu(\mathbf{x}, t + t_1)}{\beta_0} \hat{T}'_{0\mu}(\mathbf{x}, x_0 + t + t_1) \right] \right\}. \end{aligned} \quad (3.5)$$

Here the chemical potential is to be considered. After a reduction (3.3), term with the chemical potential became part of the Hamiltonian \hat{H}' . In analogy with (3.1), it may be presented as

$$\mu(x) = \mu_0 + \mu_1(x) \quad (3.6)$$

and the part of H' dependent on μ_1 will be considered as a perturbation.

Perturbations of thermodynamical parameters are called “thermal perturbations” [21] differing from “mechanical” ones which may be considered as part of the Hamiltonian. However after reduction (3.3) of the Hamiltonian, perturbations of the chemical potential may be formally taken into account as mechanical ones. Analogous technical tricks are not possible with the temperature and it leads to the more complicated contour for the time integration [16, 17].

Thus, instead of an ordinary perturbation in the Hamiltonian V (usually characterised by the coupling constant) we have a more complicated one

$$\hat{V} \rightarrow \hat{V} - \int d^3\mathbf{x} \mu_1(x) \hat{J}^0(x) \quad (3.7)$$

dependent on both coupling constant and chemical potential deviation $\mu_1(x)$. The first term in (3.7) corresponds to the familiar vertex and the second one will be presented in the diagram as an insertion into the line.

4 Generating functional of the theory

Now a generating functional for the Green's functions may be considered

$$\mathcal{Z}[j] = \text{Tr} \left\{ \hat{\rho}_\varepsilon \text{T} e^{j\hat{\phi}} \right\} = \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} N(t_1) \text{Tr} \left\{ \hat{\rho}_0 \text{T}_C \left[e^{\beta_1 \hat{T}'} \right] \text{T} \left[e^{j\phi} \right] \right\}, \quad (4.1)$$

where

$$e^{j\hat{\phi}} = \exp \left[i \int d^4y j(y) \hat{\phi}(y) \right]. \quad (4.2)$$

$$e^{\beta_1 \hat{T}'} = \exp \left[-i \int_C d^4x \frac{\beta_1^\mu(\mathbf{x}, t + t_1)}{\beta_0} \hat{T}'_{0\mu}(\mathbf{x}, x_0 + t + t_1) \right] \quad (4.3)$$

and $N(t_1)$ is the residual part from the partition function Ξ_l .

Using the identification

$$F(\hat{\phi}) e^{j\hat{\phi}} \equiv F \left[\frac{\delta}{i\delta j} \right] e^{j\hat{\phi}} \quad (4.4)$$

we may rewrite (4.1) in the form

$$\begin{aligned} \mathcal{Z}[j] &= \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} \times \\ &\times \exp \left[-i \int_C d^4x \frac{\beta_1^\mu(\mathbf{x}, t + t_1)}{\beta_0} T'_{0\mu} \left[\frac{\delta}{i\delta j(\mathbf{x}, x_0 + t + t_1)} \right] \right] Z[j, j'] \Big|_{j'=0}, \end{aligned} \quad (4.5)$$

where $T'_{\mu\nu}[\delta/i\delta j]$ is the same function of $\delta/i\delta j$ as $T'_{\mu\nu}(\phi)$ of ϕ and

$$Z[j, j'] = \text{Tr} \left\{ \hat{\rho}_0 \text{T}_C \left[e^{j\hat{\phi}} \right] \text{T} \left[e^{j'\hat{\phi}'} \right] \right\} \quad (4.6)$$

The generating functional (4.6) may be defined with the unit time-ordering procedure if the time-integration contour \mathcal{C} (fig. 1) is introduced. This contour differs from the one considered in the previous papers [16, 17], however it satisfies all the conditions used there. Segment \mathcal{C}_1 is one where external fields fixed in the second exponent in (4.6) and $\mathcal{C}_{2''}$ and \mathcal{C}_3 — for thermal perturbations in the first time-ordered exponent in the expression (4.6). As previously described, the vertical part is considered at $-\infty$ and, therefore, it plays no role. A difference between the contour indices $2'$ and $2''$ will be observable from

the region of time integration in the generating functional and it is not specified by the separate index value of ϕ_a and J_a . This contour consists only of 3 horizontal segments and the theory is based on the 3×3 free propagators (instead 4×4 used previously [16, 17]).

It is clear that the generating functional has the same form as in [16]:

$$\begin{aligned}
\mathcal{Z}[j_1, j_2, j_3] = & \varepsilon \int_{-\infty}^0 dt_1 e^{\varepsilon t_1} \exp \left\{ -i \int d^4x \sum_{a=1}^3 (-1)^a V \left[\frac{\delta}{i\delta j_a(\mathbf{x}, x_0 + t + t_1)} \right] \right\} \times \\
& \times \exp \left\{ i \int_{-\infty}^0 dx_0 \int d^3\mathbf{x} \frac{\beta_1^\mu(\mathbf{x}, t + t_1)}{\beta_0} \sum_{a=2}^3 (-1)^a T'_{0\mu} \left[\frac{\delta}{i\delta j_a(\mathbf{x}, x_0 + t + t_1)} \right] \right\} \times \\
& \times \exp \left\{ -i \int d^4x \sum_{a=1}^3 (-1)^a \mu_1(\mathbf{x}, t + t_1) J_0 \left[\frac{\delta}{i\delta j_a(\mathbf{x}, x_0 + t + t_1)} \right] \right\} \times \quad (4.7) \\
& \times \exp \left\{ -\frac{i}{2} \int d^4z_1 d^4z_2 \sum_{a,b=1}^3 j_a(z_1) \mathcal{D}_{\beta_0 \mu_0}^{ab}(z_1 - z_2) j_b(z_2) \right\}.
\end{aligned}$$

The difference with the generating functional derived in [16] consists in the presence of the chemical potential perturbations μ_1 and in the free Green function that depends not only on the constant part of the temperature β_0 , but also on the constant part of the chemical potential μ_0 . Free propagators dependent on the chemical potential are well-known in the equilibrium case (see for example [30, 27]).

Finally I would like to mention that in the case of a constant temperature and varying chemical potential term, β_1 is absent in (3.5) and no double-time ordered product occur in (4.1). It makes it possible to consider a theory in this particular case with the ordinary contour that is suitable in the equilibrium case, where free propagators are 2×2 matrices.

It is important to mention also that this theory is free from pinch singularities. Arguments for this fact are the same as in [16, 17].

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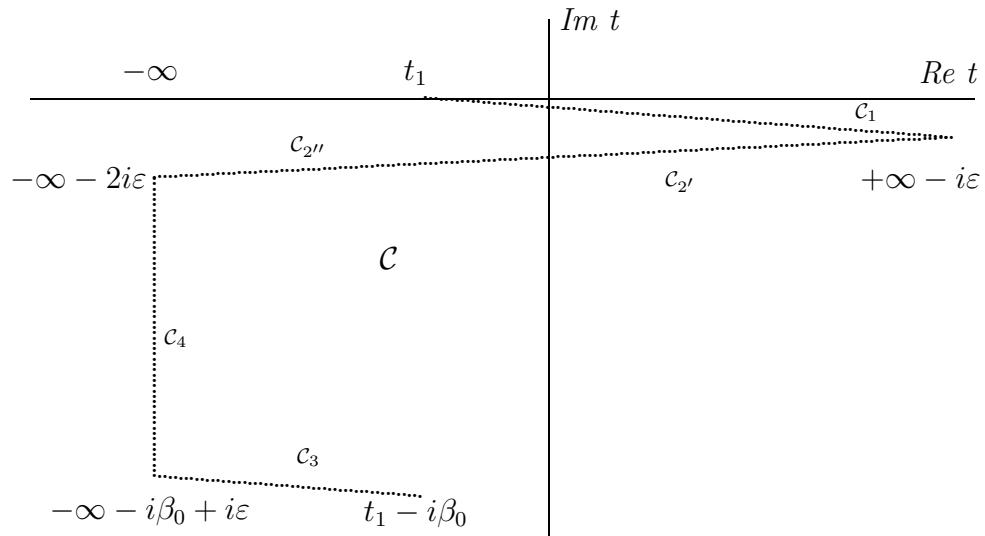


Figure 1: Contour for the nonequilibrium case

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